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NOISE PROPAGATION IN A NONUNIFORM ELECTRON GAS

TECHNICAL REPORT NO. 58

ELECTRON PHYSICS LABORATORY
Department of Electrical Engineering

By: G. Hok

Approved by: J. E. Rowe

March, 1963

CONTRACT WITH:

ELECTRONIC TECHNOLOGY LABORATORY, AERONAUTICAL SYSTEMS DIVISION, WRIGHT-PATTERSON AIR FORCE BASE, OHIO. DEPARTMENT OF THE AIR FORCE, PROJECT NO. 4156, TASK NO. 415603, CONTRACT NO. AF-33(657)-8050

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Gunnar Hok

Approved by: J. E. Rowe.
J. E. Rowe, Director
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ABSTRACT

As a first step in the development of a more general theory of propagation and transport of noise by compressional waves in a nonuniform electron gas, this report considers in particular the medium-like propagation in an electron gas in equilibrium with one or two plane emitters. In analogy with acoustic propagation in a neutral gas, adiabatic compressions are tentatively postulated; the solutions are then accepted as physically significant only if they satisfy the conditions for negligible Landau damping.

It is found that the plasma resonance frequency or low-frequency cutoff frequency required to meet these conditions occurs between infinite plane emitters only if one of the emitters is operating strongly temperature-limited. This "unsaturated state" of the gas is precisely specified by the value of an integration constant in the expression for the d-c electric field in the gas. The plasma resonance frequency is found to be constant everywhere in the gas, despite the fact that the electron density varies widely between the emitters.

In no other states, "saturated" or "supersaturated", does such a plasma frequency and the consequent approximately adiabatic propagation exist. In other words, these states show practically no real medium-like behavior.

The extension of this analysis to states of accelerated flow is in progress, but corresponding closed-form solutions of the adiabatic wave equation have not yet been found.

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NOISE PROPAGATION IN A NONUNIFORM ELECTRON GAS

I. INTRODUCTION

The problem of minimizing the noise factor of amplifiers utilizing beams of electrons requires for its solution a good understanding of the physical origin of the noise fluctuations in the beam as well as of the manner in which these fluctuations are propagated or transported along the beam. This need can be met by constructing theoretical models that are simple to handle intuitively as well as analytically and nonetheless lead to reasonably accurate predictions of the behavior of a physical device in the laboratory.

The conventional single-velocity theory, which disregards the thermal velocity spread of the electrons, fails to predict correctly the behavior of an electron gas under conditions where the drift velocity of the gas is of the same order of magnitude or smaller than the thermal velocities. In the design of more realistic and consequently more complicated models, their intuitive value will be considerably enhanced if it is possible to take advantage of analogies with previously familiar subject matter in mathematics, physics and engineering.

The exploration of such analogies is an essential facet in the work reported here. In addition to the conventional boundary-value methods and transmission-line and waveguide theory, acoustics and aerodynamics are considered, and comparison with these two fields necessarily raises thermodynamic questions. The velocity of "sound" in an electron gas, i.e., the velocity of plane compressional waves, is investigated under conditions of uniform temperature but nonuniform density. Other interesting analogies, such as possible electron shock waves, and the parallel between an electron gun and a hypersonic windtunnel have not been seriously explored.

The thermodynamics of an electron gas is approached from the known relations between the distribution function, Boltzmann's H-function and the entropy. The calculation of the entropy contribution from the distribution in velocity space is well known; on the other hand, the evaluation of the contribution from the randomness in x-y-z space proceeds along somewhat unconventional lines, although it is a natural extension of the conventional application of Poisson statistics to the shot-noise current in temperature-limited electron flow. At first sight it may appear that the electric energy associated with the x-y-z distribution is so small that the corresponding entropy contribution can be neglected. However, a nonrandom distribution could lead to considerable electric energy storage, and inclusion of this entropy component makes it possible to exclude automatically such nonphysical distributions or variations of distributions as incompatible with the laws of thermodynamics.

If thermodynamics is disregarded and an analysis of the properties of the electron gas is based solely on continuity in phase space, Lorentz force equation for the electron, and Maxwell's equations for the electromagnetic field, no unique solutions for the propagation of transverse and compressional waves are obtained. The work of Landau[1], Bohm and Gross[2], Bernstein[3], etc. on thin plasmas with a Maxwellian velocity distribution apply also to an electron gas. Their solution of the above equations by a Laplace-transform method from specified initial or boundary conditions permits a classification of component solutions into incoherent or disorganized disturbances and organized or medium-like disturbances. The former are similar to the random motions in a very thin gas of neutral particles, the latter are "organized", i.e., given coherent shape by the combined influence of the electromagnetic field and the stationary electron statistics. The latter solutions are of two different kinds, one essentially electromagnetic waves (TEM, TM, and TE waves) with the dielectric constant and the velocity of propagation modified by the presence of the electrons, the other purely compressional or "acoustic" in its nature. The solutions of primary importance in electron devices are the TM-waves, which are necessarily at least two-dimensional. However, a one-dimensional solution of the compressional kind, which is much simpler to handle analytically, permits certain conclusions about corresponding TM-solutions, in the present case as well as in the well established single-velocity approximation. Landau showed that the organized compressional waves are unattenuated only as an asymptotic limit. The "Landau damping", however, is very small in many problems of technological interest. Like copper losses in short waveguides or waveguide components, it can often be disregarded in a first-order theory.

The approximately undamped compressional solutions which are characteristic of the electron gas as an "elastic" medium, may be determined by borrowing from the conventional derivation of the acoustic wave equation the idea that the expansions and contractions of the gas should be adiabatic, i.e., leave its entropy invariant. In acoustics this postulate disregards the presence of viscous friction; in an electron gas it obviously excludes Landau damping. This idealization of the physical relations suggests that the physical significance of the result has to be evaluated in each case. If the solutions have wavelengths large compared to the Debye wavelength, the Landau damping is known to be small and the solutions are good approximations. On the other hand, conditions indicating heavy damping suggests that the gas has no appreciable medium-like behavior. Correlations from point to point or from instant to instant are insignificant.

An adiabatic change is reversible; it is easily understood that an irreversible change of the velocity distribution will take place if the density in velocity space is appreciable in the vicinity of a velocity equal to the phase velocity of the electric field. The requirement quoted above is of course equivalent to the demand that the phase velocity of the wave be far outside the thermal velocity spread of the electrons.

The same result as obtained by means of the adiabatic condition can be achieved by another postulate, which is arrived at by a completely different reasoning, presented in a previous report[4,5]. The formulation of this postulate involved the statement of the electrokinetic power flow in the form of the product of a convection-current density and an electrokinetic potential. It was shown that such an equivalent potential may exist in a one-dimensional environment. The criterion for small medium-like disturbances was taken to be that a small plane-wave perturbation should be accompanied by small but nonzero perturbations of convection-current density and equivalent potential; for perturbations of other kinds no finite equivalent potential exists for a nonzero convection-current perturbation.

The equivalence of the adiabatic and the electrokinetic-potential criteria will be tested in the appropriate contexts in the present investigation.

II. THEORETICAL POSTULATES AND OUTLINE OF PROCEDURE

Let $\rho(r,u)$ be the expected value of the statistical number density of electrons in an electron gas at the point (x,y,z,u_x,u_y,u_z) in μ-space. Continuity in μ-space and the Lorentz force equation then combine to the following form of Boltzmann's transport equation

$$\frac{\partial\rho}{\partial t} + (\vec{u}\cdot\vec{\nabla}_r)\rho - \eta(\vec{E} + \vec{u} \times \vec{B})\cdot\vec{\nabla}_u\rho = \left(\frac{\partial\rho}{\partial t}\right)_{coll.} \cong 0 \;, \qquad (1)$$

where the differential vector operators

$$\nabla_r = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \;, \qquad (2)$$

$$\nabla_u = i\frac{\partial}{\partial u_x} + j\frac{\partial}{\partial u_y} + k\frac{\partial}{\partial u_z} \qquad (3)$$

and the constant

$$-\eta = -\frac{q_e}{m_e} \qquad (4)$$

is the charge-to-mass ratio of the electron. All electron velocities are assumed to be so small compared to the velocity of light that the relativistic mass variation can be disregarded. The last two members of Eq. 1 indicate that discrete binary interactions, "collisions", are assumed to be a negligible factor.

If Eq. 1 is integrated over the full range of the velocity coordinates, the result states conservation of electrons in (x,y,z)-space. If all terms are multiplied by mu before the integration, conservation of momentum is expressed by the equation obtained; finally, the factor $mu^2/2$ leads to a statement of conservation of energy. Conservation of higher-order moments of the velocity distribution is of less interest.

In a previous report[4,5] the third one of the above mentioned integrated equations was used in conjunction with Maxwell's equations to find the appropriate form of Poynting's theorem for a volume containing an electron gas

$$\oint_V \operatorname{div}\,(\vec{P}_{em} + \vec{P}_{ek})\,dV = \oint_S (\vec{P}_{em} + \vec{P}_{ek})\,d\vec{S} = \oint_V \frac{\partial}{\partial t}(\mathcal{E}_{em} + \mathcal{E}_k)\,dV \quad , \tag{5}$$

where $\mathcal{E}_{em}$ and $\mathcal{E}_k$ are the electromagnetic and kinetic energy density, respectively, and P_{em} and P_{ek} denote electromagnetic and electrokinetic power flow, respectively.

For periodic disturbances the last member averages out to zero; consequently, the theorem states that under steady-state periodic (or stationary stochastic) conditions the net average power crossing any closed boundary surface is zero.

The particularly interesting quantity in Eq. 5 is the electrokinetic power flow

$$P_{ek} = \frac{1}{2} m_e \left[\sum_{x,y,z} \frac{\partial}{\partial x} \left\{ (\overline{u_x^3} - \overline{u_x^2} \cdot \overline{u_x})\rho_r \right\} + \operatorname{div}(\bar{u} \cdot \overline{u^2}\rho_r) \right] \quad . \tag{6}$$

In one-dimensional applications, such as plane compressional wave motion in a uniform gas, this expression reduces to

$$P_{ek} = \frac{1}{2} m_e \int_{-\infty}^{+\infty} u^3 \rho(u,z)\,du \quad .. \tag{7}$$

If a conservative elastic medium is to be used as a model for the propagation of disturbances in an electron gas, it can be shown that it must be possible to factor this vector into a product of one convection-current-density vector and a scalar "electrokinetic potential". This approach was discussed and illustrated in the previous report mentioned above.

Here we shall base the investigation of medium-like propagation on more fundamental considerations. It is helpful to compare the derivation of the D'Alembertian wave equation in acoustics. For that case the physical postulates are the equations of continuity, Newton's equation of motion, and the relation for adiabatic compression of the gas. Our Eq. 1 above combines the first two of the corresponding acoustic equations, but Maxwell's equations for the electromagnetic field do not constitute a complete counterpart to the third one. It may not be obvious if and how an adiabatic compression of an electron gas can be defined. In acoustics, an adiabatic change of state is reversible and leaves the entropy of the gas invariant. It may appear questionable that a volume or a volume element of an electron gas can be taken as a "temporarily closed system" for the purpose of evaluating thermodynamic coordinates, such as temperature, entropy, etc., from thermodynamic principles. However, statistical mechanics offers unambiguous definitions of temperature and entropy in terms of the velocity variance and the Boltzmann H-function, as long as a statistical distribution function can be stated for each volume element of the electron gas. In the ranges of frequency and temperature to be considered in this paper

$$\frac{\hbar\omega}{kT} \ll 1 \;, \qquad (8)$$

where $\hbar$ is Planck's constant h divided by 2π, ω is the radian frequency of a perturbation considered, k is Boltzmann's constant and kT is the variance of the velocity distribution in each dimension multiplied by the electron mass. Consequently classical statistical mechanics is applicable, and it will be assumed that an equilibrium state at the surface of an electron emitter or, if so stated, elsewhere, is described by a Maxwell-Boltzmann distribution function. For an ensemble of a given number N of electrons the entropy per electron in terms of the distribution density function $\hat{\rho}(r,u)$ in μ-space can then be written:

$$S_e = -k \int_{V_N} \frac{\hat{\rho}(r,u)}{N} \log \frac{\hat{\rho}(r,u)}{N} \, dV \quad , \tag{9}$$

where dV is the volume element in μ-space and V_N is the total volume in μ-space occupied by the N electrons.

A small variation $\delta\hat{\rho}$ of $\hat{\rho}(r,u)$ is adiabatic if the corresponding variation in entropy $N\delta S_e$ is zero.

$$0 = N\delta S_e = -Nk \int_{V'_N} \left[1 + \log \frac{\hat{\rho}(r,u)}{N} \right] \frac{\delta\hat{\rho}}{N} \, dV \quad , \tag{10}$$

V'_N being the volume after the adiabatic change of state.

If the position and the velocity of an electron at a given instant are taken to be independent random variables, $\hat{\rho}(r,u)$ may be written as N times the product of two independent probability densities, one in velocity space, related to the kinetic energy of an electron and one in (x,y,z) space, related to its potential energy. Because of the logarithmic nature of Eq. 9, the entropy variations may be computed separately from the two probability densities $\hat{\rho}_r$ and $\hat{\rho}_u$ and the result added

$$0 = N\delta S_e = (\delta S_r + \delta S_u)\, N \quad . \tag{11}$$

In terms of time and space coordinates this condition may be written

$$0 = N\,\frac{dS_e}{dt} = N\left\{\frac{\partial S_e}{\partial t} + (u_m \cdot \nabla)\, S_e\right\} , \tag{12}$$

where u_m is the ensemble mean of the instantaneous electron velocity. In the case of a small time-varying perturbation $\rho_1(r,u)$ on a steady-state distribution $\rho_o(r,u)$, the first-order relation between the corresponding entropy quantities obtained from Eq. 12 is

$$0 = N\left\{\frac{\partial S_{e1}}{\partial t} + (u_{mo} \cdot \nabla)\, S_{e1} + (u_{m1} \cdot \nabla)\, S_{eo}\right\} . \tag{13}$$

The complete system of equations consisting of a first-order perturbation version of Eq. 1, Maxwell's equations for the electromagnetic field and Eq. 13 determine the medium-like propagation of disturbances in an electron gas, subject to the reservation for Landau damping given previously.

For the representation of plane compressional waves μ-space may be reduced to two dimensions, u_z and z, if the z-axis is taken as the direction of propagation and the electron gas is uniform at least with x and y. Also the u_z-dimension may conveniently be eliminated by the three successive integrations of Eq. 1 mentioned above. However, it is expedient to modify this procedure slightly. Equation 1 may before integration in turn be multiplied by each Hermite polynomial $H_n(v)$ of the normalized velocity $v = u_z/\sigma$, where σ^2 is the mean square value of u_z according to the steady-state distribution at some convenient reference point. Let us define

$$a_n(z) = \frac{1}{n!} \int_{-\infty}^{+\infty} H_n(v)\rho(z,\sigma v)\sigma \, dv \quad . \tag{14}$$

Then quantities a_n may be considered as the coefficients in a series expansion of $\rho(u)$ in velocity space

$$\rho(\sigma v)\sigma dv = \sum_{n=0}^{\infty} a_n \varphi_n(v) \, dv \quad , \tag{15}$$

where

$$\varphi_n(v) = \frac{1}{\sqrt{2\pi}} H_n(v) \exp\left(-\frac{1}{2} v^2\right) = H_n(v)\varphi_o(v) \quad . \tag{16a}$$

The Hermite functions (13) are related by the equation

$$\varphi_n(v) = (-1)^n \frac{d^n}{dv^n} \varphi_o(v) \tag{16b}$$

and the polynomials by the recurrence equation

$$v \cdot H_n(v) = H_{n+1}(v) + nH_{n-1}(v) \quad . \tag{16c}$$

The sequence of integrations of Eq. 1 referred to above is equivalent to introducing Eq. 15 into Eq. 1; because of the orthogonality of the polynomials with respect to the weighting function $\varphi_o(v)$, one independent equation is obtained for each function $\varphi_n(v)$. The result is an infinite system of partial differential equations between the coefficients a_n as functions of z and t and the electromagnetic field. The latter may be eliminated simply by the use of Poisson's equation, if the scope of the analysis is limited to plane compressional waves and the magnetic field is taken to be zero or parallel to the z-axis.

$$\frac{\partial E}{\partial z} = -\frac{q_e}{\epsilon_o} \cdot \int_{-\infty}^{+\infty} \rho(u,z)\, du = -\frac{q_e}{\epsilon_o} a_o = \frac{q_e}{\epsilon_o} \frac{\partial e_o}{\partial z} . \qquad (17)$$

The symbol

$$- e_o = \int a_o dz = -\frac{\epsilon_o}{q_e} E \qquad (18)$$

is more convenient to use in the resulting differential equations than the integral of a_o.

Thus the first coefficient in the expansion (15) is proportional to the charge density in real space. The second coefficient is related to the convection current density by

$$J = - q_e \int_{-\infty}^{+\infty} u\rho(u)du = - q_e \sigma a_1 . \qquad (19)$$

The significance of the coefficients a_2 and a_3 will be explored below in the analysis of particular examples of propagation of compressional waves in an electron gas.

Although only the first three equations in the infinite nonlinear system obtained above will be used in the subsequent analysis, a brief discussion of the infinite system is in order. Before any additional condition such as Eq. 13 is imposed on the gas, the general solution has an infinite number of independent parameters; this is reasonable, since arbitrary boundary conditions or initial conditions in μ-space can be satisfied only by such a solution. However, there may be restrictions on the ensemble of solutions that can even represent a physical situation. The condition given by Eq. 13 is too restrictive; other disturbances than adiabatic ones are certainly conceivable if not indispensable. But some milder restrictions related to the second law of thermodynamics must apply to the whole physical system, including both the electron gas and its environment.

It is interesting to note that the transport equation (1) treats the electron density as a smooth continuous function. Only the ratio of charge to mass occurs, but no reference to the finite quanta of charge and mass or to the finite number of degrees of freedom of the gas. The continuous function is of course a consequence of the fact that the electron density is a probability density rather than a real discrete particle density. The parameters of the equilibrium-distribution density function are functions of the discrete particle dimensions, however, so that these dimensions do actually implicitly enter the Eq. 1.

III. THE UNIFORM NEUTRALIZED ELECTRON PLASMA

The first case of medium-like propagation to be considered is in a uniform Maxwellian electron gas at rest. In order to eliminate any steady electric field it is convenient to postulate simultaneously a uniform positive gas component of equal charge density but with such a small charge-to-mass ratio that it may be taken to remain completely at rest during the transmission of compressional electron waves. This is the case of a perfect electron plasma, the solution of which has been extensively studied. It is reviewed here as a simple illustration of the analytical procedures to be used in this report and for the purpose of discussing in more detail a few aspects of the physical phenomena involved.

The electron distribution density in velocity space is at rest independent of t and z

$$\rho_o(z,u)\,du = a_{oo}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\frac{u^2}{\sigma^2}\right)\frac{du}{\sigma}$$

$$= a_{oo}\,\varphi_o(v)dv = -\frac{\partial e_o}{\partial z}\,\varphi_o(v)dv \quad . \tag{20}$$

Let a small perturbation of ρ_o be denoted by

$$\rho_1(t,z,u)du = \sum_{n=0}^{\infty} a_{1n}\varphi_n(v)dv \quad . \tag{21}$$

It shall be assumed that the perturbation is so small that the system of equations obtained from Eqs. 1 and 18 by setting

$$\rho(u) = \rho_o(z,u) + \rho_1(t,z,u) \tag{22}$$

is very nearly linear in terms of the perturbation ρ_1. In case of pure compressional waves a sufficient condition for linearity is that for any u_1 and u_2

$$\int_{u_1}^{u_2} \rho_1 du \ll \int_{-\infty}^{+\infty} \rho_o du \quad . \tag{23}$$

If solenoidal disturbances are included a more general criterion may be borrowed from the acoustics of solids: the periodic excursions of the electrons from their unperturbed paths must be negligible compared to the wavelength or attenuation length of the vibration.

The infinite system of first-order equations then is

$$-\frac{\partial^2 e_{10}}{\partial t \partial z} + \sigma \frac{\partial a_{11}}{\partial z} = 0 \quad , \tag{24.1}$$

$$\frac{\partial a_{11}}{\partial t} - \sigma \frac{\partial^2 e_{10}}{\partial z^2} + 2\sigma \frac{\partial a_{12}}{\partial z} - \frac{\eta q_e}{\epsilon_o \sigma} e_{10} \frac{\partial e_{00}}{\partial z} = 0 \quad , \tag{24.2}$$

$$\frac{\partial a_{12}}{\partial t} + \sigma \frac{\partial a_{11}}{\partial z} + 3\sigma \frac{\partial a_{13}}{\partial z} = 0 \quad , \tag{24.3}$$

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$$\frac{\partial a_{1n}}{\partial t} + \sigma \frac{\partial a_{1n-1}}{\partial z} + (n+1)\, \sigma \frac{\partial a_{1n+1}}{\partial z} = 0 \quad . \tag{24.n}$$

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The assumption of a neutralizing positive space charge leads to an apparent inconsistency. The symbol

$$- e_{oo} = \int a_{oo} dz$$

from Eq. 18 above has been used as the integral of the <u>electron</u> density, although the integral of the total charge density is zero. Thus the last member of Eq. 18 does not apply to the d-c field component, which is zero.

Clearly the system in Eq. 24 is indeterminate, since any n equations contain n+1 dependent variables.

Let us now restrict the discussion to adiabatic perturbations. Then an additional equation may be obtained from Eq. 11 in the previous section. It is not necessary to use Eq. 13 here, since the unperturbed gas has zero mean velocity u_{mo} and its entropy S_{eo} is invariant with z. The entropy and entropy perturbation per electron derived from the velocity distribution ρ_o and the perturbation $\delta\rho$ are then

$$S_u = -k \int_{-\infty}^{+\infty} \frac{\rho_o(z,u)}{a_{oo}} \log \frac{\rho_o(z,u)}{a_{oo}} du$$

$$= -k \log \sqrt{2\pi e \sigma^2} \quad \text{for Maxwellian gas} \quad , \tag{25a}$$

$$\delta S_u = -k \int_{-\infty}^{+\infty} \left[1 + \log \frac{\rho_o(z,u)}{a_{oo}}\right] \frac{\delta\rho(z,u)}{a_{oo}} du \quad , \tag{25b}$$

where

$$a_{oo} = \int_{-\infty}^{+\infty} \rho_o(z,u)du = N \quad . \tag{26}$$

Since the number of electrons in the group remains constant during an adiabatic change of state, it is required that

$$\int_{-\infty}^{+\infty} \delta\rho(z,u)du = 0 \quad . \tag{27}$$

The perturbation per unit volume $\rho_1(z,u)$ given by Eq. 21 obviously cannot satisfy this requirement. Since infinitesimal readjustment of the average density affects only the first term in the expansion (21) appreciably, the variation $\delta\rho$ to be used in Eq. 25 is simply the right-hand side of Eq. 21 with the first term omitted. Consequently

$$\delta S_u = -k \int_{-\infty}^{+\infty} \left[1 - \frac{1}{2}\log 2\pi - \frac{1}{2}v^2\right] \cdot \sum_{n=1}^{\infty} a_{1n}\varphi_n(v)dv = ka_{12} \quad . \tag{28}$$

The potential energy associated with the z-coordinate of each electron is much smaller than $kT/2$; at first sight it may therefore appear that the z-distribution of the electrons can be neglected in the computation of the entropy variation. However, the potential fluctuations represent the only collective and individual interaction in the gas and are essential to its medium-like properties. Furthermore, if the distribution of electrons in space were not random, the potential energy could easily become very large.

The statistics of the potential fluctuations is not explicitly needed for the entropy calculation, only the probability density of the z-coordinates of the electrons or of some equivalent generalized coordinates. It is convenient to choose the differences between the z-coordinates of neighbor electrons and to assume complete randomness, i.e., that the number of electrons per volume element has a Poisson distribution. Then this one-dimensional "distance" ℓ between electrons has the probability density

$$P(\ell)\, d\ell = \exp(-\ell/L)\, d\ell/L \qquad [0 < \ell < \infty] \quad . \tag{29}$$

It can easily be shown by conventional variational procedure that this probability density maximizes the z-component of the entropy under the constraint of given expected value of the electron density $1/L = a_{oo}$.

This entropy component of a group of N electrons is consequently

$$NS_r = - Nk \int_0^\infty P \log P \, d\ell = Nk \log Le \quad , \qquad (30a)$$

where e is the base of the natural logarithms.

The entropy variation due to a perturbation $\delta P(\ell)$

$$N\delta S_r = - Nk \int_0^\infty \left[1 - \log L - \frac{\ell}{L} \right] \delta P(\ell) \, d\ell$$

$$= Nk \int_0^\infty \frac{\ell}{L} \delta P(\ell) \, d\ell = Nk \frac{\delta L}{L} = - Nk \frac{a_{10}}{a_{00}} = - ka_{10} \quad . \qquad (30b)$$

The third and fourth members of Eq. 30b express N times Boltzmann's constant multiplied by the variation of the expected value of the one-dimensional electron "distance", divided by the unperturbed mean distance L. The fifth member follows from the fact that an infinitesimal increase in the relative mean distance in one dimension is identical to the decrease in the relative electron density if the density is invariant in the two orthogonal dimensions.

The total entropy per electron in one dimension of a Maxwellian electron gas with a Poisson distribution in space is obtained from Eqs. 25a and 30a.

$$S_e = S_u + S_r = k \log \sqrt{2\pi e \sigma^2} + k \log (eL)$$

$$= k \log (\sigma L) + \text{constant}$$

$$= k \log \left(\frac{\sigma}{a_{00}} \right) + \text{constant} \quad . \qquad (31a)$$

Consequently an adiabatic change of state requires that

$$\frac{\sigma^2}{a_{oo}^2} = \frac{kT}{m}\,\frac{V_N^2}{N^2} = \text{constant} \quad , \tag{31b}$$

where V_N is the volume occupied by N electrons and T is the absolute temperature. In this relation the pressure may be introduced in order to permit a comparison with the gas laws:

$$p = 2\int_0^\infty mu^2\rho(u)du = m\sigma^2 a_{oo} = kT\frac{N}{V_N} \tag{32}$$

giving the result

$$\rho\, V_N^3 = \text{constant} \tag{31c}$$

which is to be compared with the classical relation for adiabatic variations

$$\rho\, V_N^{\gamma_c} = \text{constant} \quad , \tag{31d}$$

where γ_c is the ratio of the specific heat at constant pressure to that at constant volume. That this ratio is equal to 3 under conditions where a single degree of freedom per gas particle is involved is well known.

The general result for a small adiabatic perturbation in a uniform electron gas at rest is according to Eqs. 25b and 30b.

$$0 = N(\delta S_u + \delta S_r) = k(a_{12} - a_{10}) \tag{33a}$$

or

$$a_{12} = a_{10} \quad . \tag{33b}$$

From Eqs. 24.1 and 24.3 it is found that Eq. 33b is equivalent to

$$\frac{\partial a_{13}}{\partial z} = 0 \quad . \tag{33c}$$

In the previous report[4] the condition $a_{13} = 0$ for medium-like propagation in a uniform equilibrium electron gas was derived from the postulate that the perturbation power flow be expressible as the product of the perturbation convection-current density and an equivalent perturbation potential. The consistent results obtained from radically different postulates suggest that either one approach may be used alternatively whichever one happens to be more convenient in each particular case. For this reason the essential points of the equivalent-potential approach shall be reviewed.

The general definition of equivalent potential is obtained from the power flow, Eq. 7

$$\frac{1}{2} m_e \int_{-\infty}^{+\infty} u^3 \rho(u,z) du = V \int_{-\infty}^{+\infty} -q_e u \rho(u,z) du \quad . \tag{34}$$

For unperturbed equilibrium conditions both integrals are zero, and V appears indeterminate, but if the whole velocity distribution is displaced by a small velocity Δu, $V(\Delta u)$ is a continuous function of Δu and approaches a limit as Δu approaches zero. In this way the unperturbed equivalent potential, or "kinetic potential" is found to be

$$V_o = \frac{3}{2\eta} \frac{\int_{-\infty}^{+\infty} u^2 \rho_o(u) du}{\int_{-\infty}^{+\infty} \rho_o(u)\ du} = \frac{3}{2n} \sigma^2 \quad . \tag{35}$$

The first-order perturbation of the power flow may be written

$$\frac{1}{2} m_e \int_{-\infty}^{+\infty} u^3 \rho_1(u,z)\, du = V_o J_1 + V_1 J_o \quad . \tag{36}$$

Since J_o is equal to zero, V_1 can be a small perturbation only if

$$\frac{1}{2} m_e \int_{-\infty}^{+\infty} (u^3 - 2\eta V_o u)\, \rho_1(u)\, du = 0 \tag{37}$$

or

$$\frac{1}{3!} \int_{-\infty}^{+\infty} (v^3 - 3v)\, \rho_1(\sigma v)\, dv = a_{13} = 0 \quad . \tag{38}$$

In the present case, where the system of differential equations is linear with constant coefficients, all solutions are linear combinations of exponentials in z; consequently Eqs. 38 and 33 are identical.

The solutions describing the adiabatic propagation of disturbances in the electron gas is now readily determined by applying the condition of Eqs. 38 or 32 to the system of linear differential equations (24), which is homogeneous with constant coefficients. The first Eq. 24a can be integrated directly with respect to z; the constant of integration may be temporarily omitted, since it would make the system nonhomogenous. For perturbation with a time variation of exp ($j\omega t$) the Sturmian form is obtained by proper algebraic operations:

$$\frac{\partial a_{11}}{\partial z} = - \frac{j\omega}{\sigma} a_{12} \quad , \tag{39a}$$

$$\frac{\partial a_{12}}{\partial z} = - \frac{j\omega}{3\sigma}\left(1 - \frac{\omega_p^2}{\omega^2}\right) a_{11} \quad , \tag{39b}$$

where

$$\omega_p^2 = -\frac{\eta q_e}{\epsilon_o}\frac{\partial e_{oo}}{\partial z} = \frac{\eta q_e}{\epsilon_o} a_{oo} \quad . \tag{40}$$

In terms of the convection-current density I_1 and the equivalent potential V_1

$$I_1 = -q_e \sigma a_{11} \quad , \tag{41a}$$

$$V_1 = -\frac{3\sigma^2}{2\eta}\frac{a_{12}}{a_{oo}} \quad , \tag{41b}$$

the result may be considered to express the propagation on an equivalent transmission line or waveguide,

$$\frac{\partial I_1}{\partial z} = -YV_1 = -j\omega \frac{2q_e a_{oo}}{3\sigma^2 m_e} V_1 \quad , \tag{42a}$$

$$\frac{\partial V_1}{\partial z} = -ZI_1 = -j\omega \frac{m_e}{2q_e^2 a_{oo}}\left(1 - \frac{\omega_p^2}{\omega^2}\right) I_1 \tag{42b}$$

the propagation constant and characteristic impedance, respectively, being given by the relations

$$\gamma^2 = -ZY = \frac{\omega^2}{3\sigma^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right) \quad , \tag{43a}$$

$$Z_o = \sqrt{Z/Y} = \frac{\sigma\, m_e}{2q_e^2\, a_{oo}} \sqrt{3\left(1 - \frac{\omega_p^2}{\omega^2}\right)} \quad . \tag{43b}$$

From (43a) the phase velocity u_p and the group velocity u_g, respectively, of the plane-wave solutions of Eqs. 39a and 39b are obtained

$$u_p = \frac{\omega}{\gamma} = \sqrt{3\sigma^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2} , \qquad (44a)$$

$$u_g = \frac{d\omega}{d\gamma} = \sqrt{3\sigma^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} . \qquad (44b)$$

The phase velocity approaches infinity and the group velocity zero as the frequency approaches ω_p. The square root of the product of phase and group velocities is a characteristic of the gas which has similar form to the velocity of sound in a pure neutral gas

$$u_s = \sqrt{\frac{\gamma_c kT}{M}} , \qquad (45)$$

where γ_c is the ratio of specific heats and M the molecular mass. In comparison

$$\sqrt{u_p u_g} = \sqrt{\frac{3kT}{m_e}} \qquad (46)$$

for the electron gas. The appearances of the factor of 3 instead of γ_c in this expression is consistent with the relations for adiabatic expansion, Eqs. 31c and 31d.

If the unperturbed velocity distribution is not Maxwellian, the transport equation still holds, but the adiabatic condition in Eq. 32 will in general change. An estimate of the entropy variation δS_u of the velocity distribution may be obtained by introducing the expansions in Hermite functions (Eq. 15) for ρ_o and $\delta\rho$. Since the second and third coefficients a_{01} and a_{02} are made equal to zero by choice of reference frame and normalization, the first and probably largest nonequilibrium component is a_{03}. Let us first evaluate δS_u for the case that a_{03}/a_{00} is a first-order small quantity and that all higher-order components are negligible. Then the velocity entropy variation is

$$\delta S_u = -\frac{k}{a_{00}} \int_{-\infty}^{+\infty} \left\{1 + \log\left[\varphi_0(v) + \frac{a_{03}}{a_{00}}\varphi_3(v)\right]\right\} \sum_{n=1}^{\infty} a_{1n}\varphi_n(v)dv$$

$$= -\frac{k}{a_{00}} \int_{-\infty}^{+\infty} \left\{1 + \log\varphi_0(v) + \log\left[1 + \frac{a_{03}}{a_{00}} H_3(v)\right]\right\} \sum_{n=1}^{\infty} a_{1n}\varphi_n(v)dv$$

$$\cong \frac{k}{a_{00}} \int_{-\infty}^{+\infty} \left\{\frac{1}{2}\log 2\pi e + \frac{1}{2} H_2(v) - \frac{a_{03}}{a_{00}} H_3(v)\right\} \sum_{n=1}^{\infty} a_{1n}\varphi_n(v)dv$$

$$\cong k\left[\frac{a_{12}}{a_{00}} - 3!\,\frac{a_{03}\,a_{13}}{a_{00}^2}\right] . \tag{50}$$

Consequently the conditions $a_{13} = 0$ and $a_{12} = a_{10}$ are still compatible at least to the first order, and the previous results apply also to this case. For larger deviations from a Maxwellian distribution the adiabatic condition appears to become considerably more complicated.

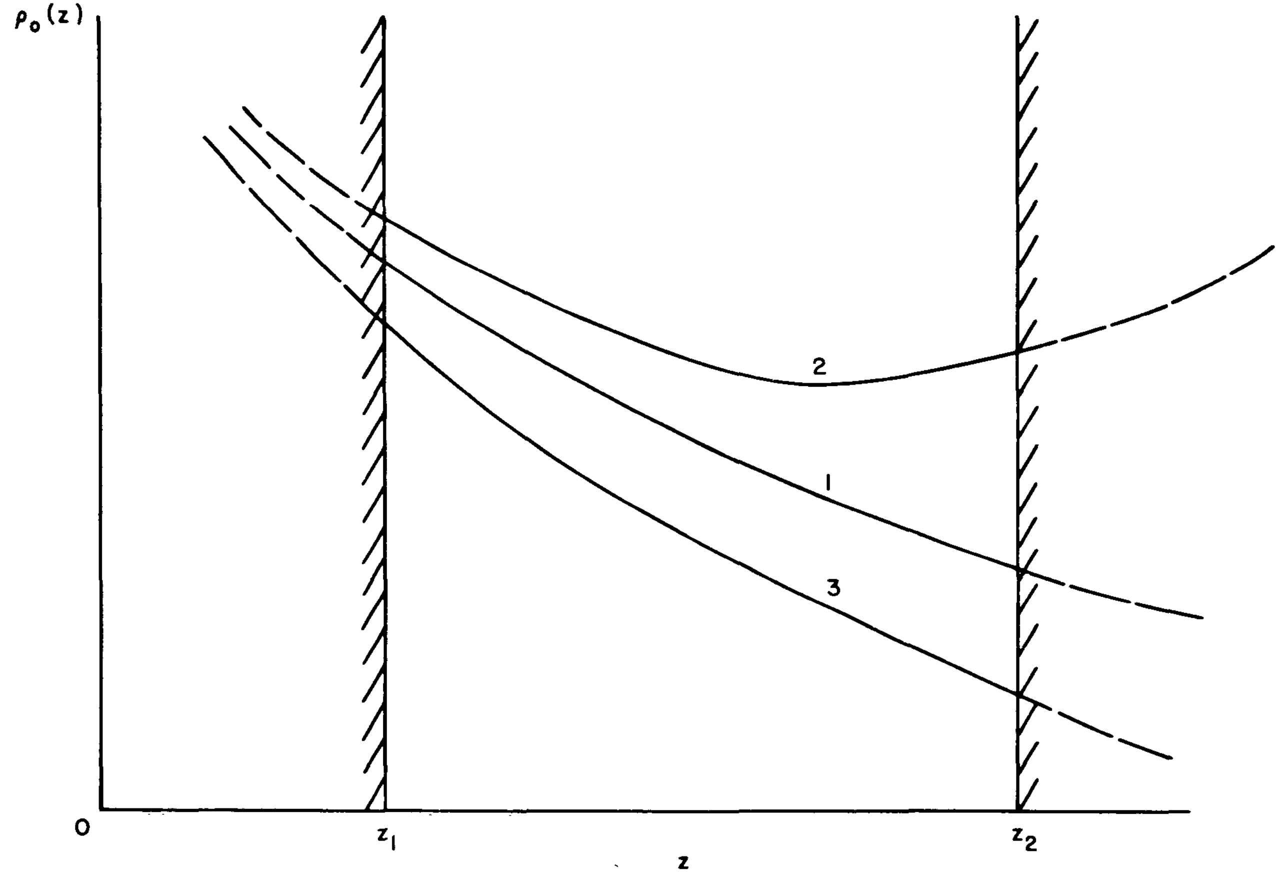
In the equivalent-potential model, the unperturbed kinetic potential appears to be infinite, according to Eq. 34, when a_{03} is not zero in the reference frame where a_{01} is zero. This is physically unreasonable and it suggests very strongly that the general representation of the electro-kinetic power flow as the product of an equivalent potential and the convection-current density breaks down under these circumstances. However, the entropy relations just presented seem to indicate that the adiabatic solution can be found by disregarding the third velocity moment, since it neither contributes to nor obstructs the medium-like behavior of the gas. In other words, the unperturbed potential V_o may be taken from Eq. 35 even when the unperturbed distribution differs somewhat from a Maxwellian distribution. Also the perturbation potential V_1 is then obtained as before from the condition $a_{13} = 0$, in full agreement with the constant-entropy approach.

The criteria of finite V_o and small V_1 thus appears to offer an equivalent alternative to the adiabatic condition for finding medium-like solutions in an electron gas. The next section of this report will discuss a case of a nonuniform electron gas in which it can be established that the constant-entropy approach and the equivalent-potential approach lead to exactly the same set of differential equations.

IV. ELECTRON GAS IN EQUILIBRIUM WITH ONE OR TWO PLANE EMITTERS

4.1 The Unperturbed State

The heavy positive fluid postulated in the previous case is eliminated in the present problem; conditions of statistical equilibrium are still assumed. There is a net negative space charge and a d-c electric field, both varying in one dimension only, i.e., the z-dimension. The average net current density is zero everywhere, and the electron velocity distribution is Maxwellian with a temperature equal to the temperature T of the plane electron emitters at z_1 and z_2 (Fig. 1). The unperturbed state of the gas is determined from the time-invariant components of the variables in Boltzmann's transport equation (1) and Poisson's Eq. 17



1. "Saturated" Gas
2. "Supersaturated" Gas
3. "Unsaturated" Gas

FIG. 1. QUALITATIVE GRAPH OF THE ELECTRON DENSITY BETWEEN THE EMITTERS.

$$\rho_o(z,u)\,du = a_{oo}(z)\varphi_o(u/\sigma)\,du/\sigma \quad , \tag{51}$$

$$\frac{\epsilon_o}{q_e} E_o(z) = e_{oo}(z) \quad , \tag{52}$$

giving the two equations

$$\frac{\partial a_{oo}}{\partial z} + 2\theta e_{oo} a_{oo} = 0 \quad , \tag{53}$$

$$-\frac{\partial e_{oo}}{dz} = a_{oo} \quad , \tag{54}$$

where the only remaining parameter is

$$\theta = \frac{q_e\,\eta}{2\epsilon_o \sigma^2} \quad . \tag{55}$$

Elimination of a_{oo} leads to the equation

$$\frac{\partial^2 e_{oo}}{\partial z^2} + 2\theta e_{oo} \frac{\partial e_{oo}}{\partial z} = \frac{\partial^2 e_{oo}}{\partial z^2} + \theta \frac{\partial}{\partial z} (e_{oo}^2) = 0 \quad . \tag{56}$$

After one integration and separation of the variables, the results are, respectively,

$$\frac{\partial e_{oo}}{\partial z} + \theta\,(e_{oo}^2 + C_1^2) = 0 \tag{57a}$$

and

$$\frac{de_{oo}}{e_{oo}^2 + C_1^2} = \frac{1}{C_1} \frac{d(e_{oo}/C_1)}{(e_{oo}/C_1)^2 + 1} = -\theta dz \quad . \tag{57b}$$

Depending on the value of the integration constant C_1^2, three different types of solutions may be obtained. The simplest form results if

$$C_1^2 = 0 \quad . \tag{58a}$$

Then

$$e_{oo} = \frac{1}{\theta z} \quad , \tag{58b}$$

$$a_{oo} = \frac{1}{\theta z^2} \quad . \tag{58c}$$

The second integration constant determines the origin on the z-axis. This constant is zero, if the origin is chosen at the plane where an emitter of infinite electron density would be located.

Both e_{oo} and a_{oo} are in this case monotone decreasing functions of z. The intermediate curve in Fig. 1 indicates qualitatively the variation of the electron density.

In the case

$$C_1^2 > 0 \tag{59a}$$

the corresponding integrals are

$$\frac{1}{C_1} \tan^{-1}\left(\frac{e_o}{C_1}\right) = -\theta z + C_2 = \frac{1}{C_1}\left(\frac{\pi}{2} - gz\right) \quad , \tag{59b}$$

$$e_{oo} = C_1 \cot gz \quad , \tag{59c}$$

$$a_{oo} = \frac{C_1 g}{\sin^2 gz} \quad , \tag{59d}$$

where

$$g = \theta C_1 = \frac{q_e \eta C_1}{2\epsilon_0 \sigma^2} , \tag{59e}$$

where the second integration constant has been chosen so that the origin of z may be located as in the first solution. In this case the electron density has a minimum, as indicated by the top curve in Fig. 1. The location of this minimum is

$$z_{min} = \frac{\pi}{2g} . \tag{59f}$$

A second singularity occurs at

$$z_{2\infty} = \frac{\pi}{g} , \tag{59g}$$

which is the location of the right-hand emitter, if its electron density is infinite.

If, finally,

$$C_1^2 = - C_3^2 < 0 , \tag{60a}$$

the results are

$$e_{oo} = C_3 \coth gz , \tag{60b}$$

$$a_{oo} = \frac{C_3 g}{\sinh^2 gz} , \tag{60c}$$

where now

$$g = \theta C_3 = \frac{q_e \eta C_3}{2\epsilon_0 \sigma^2} . \tag{60d}$$

The electron density is a monotone decreasing function, indicated by the bottom curve in Fig. 1.

Which one of these three different charge distributions is applicable in a particular situation may be determined from appropriate boundary conditions at the emitters. At a given emitter temperature and given emitter density $a_{oo}(z_1)$ on the left there are three parameters, only two of which may be chosen independently, if equilibrium is to be maintained. These parameters are: (1) the distance between emitters $z_2 - z_1$, (2) the ratio of the electron densities at the emitters $a_{oo}(z_2)/a_{oo}(z_1)$, and (3) the potential difference between the emitters. For a given interemitter distance there is only one value of $a_{oo}(z_2)$ that will result in the first case described by the Eq. 58. A better right-hand emitter gives conditions described by Eq. 59, and a poorer right-hand emitter leads to the state in Eq. 60.

The first state may be generated by removing the right-hand emitter to infinity letting the left-hand emitter reach an equilibrium condition in relation to its own emission into semi-infinite space. Insertion of a second emitter which <u>increases</u> the electron density gives the "supersaturated" state of Eq. 59, while a second emitter which <u>decreases</u> the density gives the "unsaturated" state of Eq. 60. In each case the emitters are insulated, so that they assume the potential difference required to make the average current zero. It may be noted that the distinction between the supersaturated and unsaturated states is not in general the same as the distinction between space-charge-limited and temperature-limited operation of the right-hand emitter (the lower-potential or lower-emission electrode). These criteria coincide only in the limit as the position of the right-hand emitter approaches infinity, $z_2 \to \infty$. For the case $C_1^2 > 0$ the right-hand emitter will operate temperature-limited, if it is placed between the density minimum and the left-hand emitter.

4.2 Propagation of Perturbations

The first-order perturbation equations derived from Eqs. 1 and 17 differ from the system of Eq. 24 in two respects. Since the d-c electric field is not zero, another term will appear in the second equation. Furthermore, the system of equations now has variable coefficients, since both e_{00} and $\partial e_{00}/\partial z$ are functions of z.

The first-order system of equations is

$$-\frac{\partial^2 e_{10}}{\partial t\,\partial z} + \sigma \frac{\partial a_{11}}{\partial z} = 0 \ , \qquad (61a)$$

$$\frac{\partial a_{11}}{\partial t} - \sigma \frac{\partial^2 e_{10}}{\partial z^2} + 2\sigma \frac{\partial a_{12}}{\partial z} - 2\sigma\theta \left(e_{10} \frac{\partial e_{00}}{\partial z} + e_{00} \frac{\partial e_{10}}{\partial z} \right) = 0 \ , \qquad (61b)$$

$$\frac{\partial a_{12}}{\partial t} + \sigma \frac{\partial a_{11}}{\partial z} + 3\sigma \frac{\partial a_{13}}{\partial z} + 2\sigma\theta e_{00} a_{11} = 0 \ . \qquad (61c)$$

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The first equation is the same as Eq. 24a and can be integrated directly. Like the system Eqs. 24, Eq. 61 has too many dependent variables to be determinate. The velocity distribution is still Maxwellian everywhere. The distribution of electrons with z is not invariant with z; however, at any fixed z the random distribution of the interelectron distance may still be assumed to have the form of Eq. 29, so that the entropy component S_r from Eq. 30a applies also in the present case. However, the mean distance L and the electron density are functions of z; consequently the adiabatic condition must be derived from Eq. 13. The mean unperturbed velocity u_{om} is still zero; so is the space variation of the velocity entropy S_{uo}. But the mean velocity perturbation is

$$u_{1m} = \frac{\sigma\, a_{11}}{a_{oo}} \tag{62}$$

and

$$\frac{\partial S_{ro}}{\partial z} = k \frac{1}{L} \frac{\partial L}{\partial z} = -k \frac{1}{a_{oo}} \frac{\partial a_{oo}}{\partial z} \quad . \tag{63}$$

If common factors k, N, $1/a_{oo}$ are omitted, Eq. 13 gives the relation

$$0 = \frac{\partial}{\partial t}(a_{12} - a_{10}) - a_{11} \frac{\partial a_{oo}}{\partial z} \frac{1}{a_{oo}}$$

$$= \frac{\partial}{\partial t}\left(a_{12} + \frac{\partial e_{10}}{\partial z}\right) + 2\theta e_{oo}\sigma a_{11} \quad . \tag{64}$$

This equation can alternatively be obtained from Eq. 61a and 61c by setting $\partial a_{13}/dz = 0$, in accord with the equivalent-potential approach.

If e_{10} is eliminated between Eqs. 61 and 64, the result may be written as a pair of transmission-line equations:

$$\frac{\partial}{\partial z}\left(\frac{a_{11}}{a_{oo}}\right) = -j\frac{\omega}{\sigma} \frac{1}{a_{oo}} a_{12} = -jB\, a_{12} \quad , \tag{65a}$$

$$\frac{\partial}{\partial z} a_{12} = -j\frac{\omega}{3\sigma} a_{oo}\left(\frac{a_{11}}{a_{oo}}\right) = -jX\left(\frac{a_{11}}{a_{oo}}\right) \quad . \tag{65b}$$

Since $(BX)^{1/2}$ is constant but $(X/B)^{1/2} = a_{oo}(z)$ varies with z, the propagation of plane compressional waves in the equilibrium electron gas is analogous to the transmission of TEM-waves along a coaxial line with constant $(\mu\epsilon)^{1/2}$ but a tapered characteristic impedance. This analogy suggests that there may be solutions of the form

$$a_{12} = h(z) \exp(-j\gamma z) \quad , \tag{66}$$

where h(z) is real and γ is a constant.

If a_{11}/a_{00} is eliminated between Eqs. 65 and the trial solution of Eq. 66 is substituted into the resulting second-order differential equation,

$$\frac{\partial^2 a_{12}}{\partial z^2} - \frac{1}{a_{00}} \frac{\partial a_{00}}{\partial z} \frac{\partial a_{12}}{\partial z} + \frac{\omega^2}{3\sigma^2} a_{12} = 0 \quad , \tag{67}$$

the following pair of equations are obtained by omitting common factors and separating real and imaginary terms

$$\frac{\partial^2 h}{\partial z^2} - \frac{1}{a_{00}} \frac{\partial a_{00}}{\partial z} \frac{\partial h}{\partial z} - \left(\gamma^2 - \frac{\omega^2}{3\sigma^2}\right) h = 0 \quad , \tag{68a}$$

$$2 \frac{\partial h}{\partial z} - \frac{1}{a_{00}} \frac{\partial a_{00}}{\partial z} h = 0 \quad . \tag{68b}$$

The second equation is separable and may be integrated directly

$$h = C_4 \sqrt{a_{00}} = C_4 \exp\left(- \theta \int e_{00} dz\right) \quad , \tag{69}$$

the last alternative being obtained from Eq. 53. Substitution of this solution into Eq. 68a yields the remaining unknown parameter γ and proves that Eq. 66 is indeed the solution of Eq. 67.

$$- \theta \left\{ \frac{\partial e_{oo}}{\partial z} - \theta e_{oo}^2 \right\} - \theta e_{oo} \cdot 2\theta e_{oo} = \left(\gamma^2 - \frac{\omega^2}{3\sigma^2} \right) = \theta^2 C_1^2 \quad . \tag{70}$$

After reduction of the first member, the last member follows from Eq. 57a. The integration constant C_1^2 is the same constant the value of which differentiates between the three different d-c charge distributions discussed earlier. Consequently γ is a constant given by the equation

$$\gamma = \pm \left(\frac{\omega^2}{3\sigma^2} + \theta^2 C_1^2 \right)^{1/2}$$

$$= \pm \frac{1}{\sigma\sqrt{3}} \left(1 - \frac{\omega_c^2}{\omega^2} \right)^{1/2} \quad , \tag{71}$$

where the critical frequency

$$\omega_c = \sqrt{3}\, \sigma (- \theta^2 C_1^2)^{1/2}$$

$$= \left\{ \frac{3\eta q_e}{2\epsilon_o} \left(\frac{\partial e_{oo}}{\partial z} + \theta e_{oo}^2 \right) \right\}^{1/2}$$

$$= \left\{ \frac{3\eta q_e}{2\epsilon_o} \left[- a_{oo} + \frac{1}{4\theta a_{oo}^2} \left(\frac{\partial a_{oo}}{\partial z} \right)^2 \right] \right\}^{1/2} \quad . \tag{72}$$

The perturbation of the convection-current density and of the electron density, respectively, may be calculated from a_{12} by reference to the Eqs. 65b, 67 and 61a. The results are

$$a_{12} = C_4 \sqrt{a_{oo}} \exp (-j\gamma z) \quad , \tag{73}$$

$$J = -\sigma q_e a_{11} = \sigma q_e \frac{3\sigma}{j\omega} \frac{\partial a_{12}}{\partial z}$$

$$= -j\omega q_e \frac{3\sigma^2 C_4}{\omega^2} \sqrt{a_{oo}} \left\{ \frac{1}{2a_{oo}} \frac{\partial a_{oo}}{\partial z} - j\gamma \right\} \exp(-j\gamma z) \quad , \qquad (74)$$

$$a_{10} = -\frac{\sigma}{j\omega} \frac{\partial a_{11}}{\partial z} = -\frac{3\sigma^2}{\omega^2} \frac{\partial^2 a_{12}}{\partial z^2}$$

$$= \frac{3\sigma^2 C_4}{\omega^2} \sqrt{a_{oo}} \left\{ \left(\frac{1}{2a_{oo}} \frac{\partial a_{oo}}{\partial z} \right)^2 - j\gamma \frac{1}{2a_{oo}} \frac{\partial a_{oo}}{\partial z} - \frac{\omega^2}{3\sigma^2} \right\} \exp(-j\gamma z) \quad . \qquad (75)$$

In the three different types of equilibrium states derived earlier, these relations take the specific form given below:

1. $\underline{C_1^2 = 0}$

$$a_{12} = \frac{A_1}{z} \exp(-j\gamma z) \quad , \qquad (76)$$

$$J = j\omega q_e \frac{A_1}{\gamma^2 z^2} (1 + j\gamma z) \exp(-j\gamma z) \quad , \qquad (77)$$

$$a_{10} = \frac{A_1}{\gamma^2 z^3} (1 + j\gamma z - \gamma^2 z^3) \exp(-j\gamma z) \quad , \qquad (78)$$

$$\gamma^2 = \frac{\omega^2}{3\sigma^2} \quad . \qquad (79)$$

2. $\underline{C_1^2 > 0}$

$$a_{12} = \frac{A_2}{\sin gz} \exp(-j\gamma z) \quad , \qquad (80)$$

$$J = j\omega q_e \frac{A_2}{\sin^2 gz} (g \cos gz + j\gamma \sin gz) \exp(-j\gamma z) \quad , \tag{81}$$

$$a_{10} = \frac{A_2}{\sin^3 gz} (g^2 + j\gamma g \cos gz \sin gz - \gamma^2 \sin^2 gz) \exp(-j\gamma z) \quad , \tag{82}$$

$$\gamma^2 = \frac{\omega^2}{3\sigma^2} + g^2 \quad , \tag{83}$$

$$g^2 = \theta^2 C_1^2 > 0 \quad . \tag{84}$$

3. $\underline{C_1^2 < 0}$

$$a_{12} = \frac{A_3}{\sinh gz} \exp(-j\gamma z) \quad , \tag{85}$$

$$J = j\omega q_e \frac{A_3}{\sinh^2 gz} (g \cosh gz + j\gamma \sinh gz) \exp(-j\gamma z) \quad , \tag{86}$$

$$a_{10} = \frac{A_3}{\sinh^3 gz} (g^2 + j\gamma g \cosh gz \sinh \gamma z - \gamma^2 \sinh^2 gz) \exp(-j\gamma z) \quad , \tag{87}$$

$$\gamma^2 = \frac{\omega}{3\sigma^2} - g^2 = \frac{\omega^2}{3\sigma^2}\left(1 - \frac{\omega_c^2}{\omega^2}\right) \quad , \tag{88}$$

$$g^2 = \theta^2 C_3^2 = -\theta^2 C_1^2 > 0 \quad . \tag{89}$$

The last case, the unsaturated state, has the greatest significance of the three, since Eq. 88 indicates that the phase velocity becomes infinite at

$\omega = \omega_c$. Consequently a frequency range exists, where the phase velocity is much larger than the thermal electron velocities and the Landau damping is negligible. In other words, the vibrations are adiabatic and the solutions found are in this range consistent with the assumption of constant entropy. The factor g appearing in the hyperbolic functions is not an attenuation constant; these functions describe the varying wave impedance of the electron gas, not an attenuation of the power propagated by the waves; that this power is conserved, (i.e., invariant with z) can easily be verified by calculation of the power flow as the real part of the product of a_{12}/a_{00} and the complex conjugate of a_{11} multiplied by an appropriate real constant.

The results in Case 3 above thus show a striking similarity to those for the uniform neutral plasma in Section III. Equation 88 is exactly the same as Eq. 43a. Despite the widely varying electron density a "plasma resonance frequency" or low-frequency cutoff frequency ω_c, invariant with the space coordinates, exists in the electron gas, related not to the electron density only, as in Eq. 40, but also to the density gradient as expressed by Eq. 72.

For the other two cases, the saturated and supersaturated states of an electron gas in equilibrium, the results of this investigation are largely of a negative character, but nonetheless of considerable interest. No plasma resonance or cutoff frequency exists in these cases. The phase velocity of the postulated adiabatic waves is equal to or smaller than the mean square electron velocity. Waves of such velocities are subject to heavy Landau damping; in other words, the solutions have no other physical significance than the demonstration that no even approximately adiabatic waves exist in an electron gas under these conditions.

The gas does not show the properties of an elastic medium but rather those of an extremely viscous liquid. Time and space correlations of fluctuations in the gas are very short. Any disturbance from an external source is rapidly attenuated. Noise is absorbed and emitted at equal rates, so that the noise level is maintained at that of equilibrium thermal noise at the temperature of the gas.

V. EXTENSION OF THE ADIABATIC-PERTURBATION INVESTIGATION TO ACCELERATED FLOW

In an electron gun the electron gas is in a state of flow, which is very different from the equilibrium states analyzed in previous sections of this report. The intuitive understanding and the mathematical analysis of noise transport or propagation in an electron gun would be considerably facilitated if the adiabatic trial-and-error approach that worked so well in the equilibrium gas could produce analogous closed-form solutions for accelerated flow, so that the existence of a plasma resonance frequency and medium-like wave propagation could be proved or disproved under various operating conditions or in various regions of the electron gun.

A certain amount of progress has been made along this path, but closed-form expressions for the adiabatic solutions corresponding to Eqs. 76 to 89 have not yet been found. The increased difficulties are of course connected with the fact that the mean velocity and acceleration are no more zero and the velocity variance (or the "temperature") no more constant throughout the gas. The expansion of the unperturbed flow is not completely reversible, i.e., adiabatic, everywhere, because of the sorting of the electrons into one class that passes the potential minimum and another class that does not. Past the potential minimum the expansion becomes reversible, as long as the gas is not compressed beyond the density at the potential minimum, i.e., again reaches a potential as low as the potential at the minimum.

The electron distribution in velocity space is now expressed in Hermite functions of the velocity variable

$$v = \frac{u - u_{mo}}{\sigma} ,$$

where u_{mo} is the mean unperturbed velocity and σ^2 is the variance of the velocity in the unperturbed state. Both u_{mo} and σ are functions of z.

The first-order perturbation equations obtained from Boltzmann's and Poisson's equations, combined with the condition for adiabatic perturbations, can be brought to the form of a pair of transmission-line equations

$$\frac{\partial V}{\partial z} = - jX(z)I ,$$

$$\frac{\partial I}{\partial z} = - jB(z)V ,$$

where V and I are functions of the coefficients a_{10}, a_{11}, a_{12} and the parameters of the unperturbed stream; X and B are real functions of the latter.

No closed-form solution of this system of equations has been found as yet. Already the unperturbed state is of a rather complicated transcendental nature for space-charge-limited flow in an electron gun. Without such a solution it is difficult to ascertain whether or not a cutoff frequency or plasma resonance frequency exists, and if so, how it varies with the different parameters of the gun. The final condition in a beam, drifting at high velocity in an approximately field-free space, is well established: here a plasma resonance frequency exists, and the magnitude of the noise power in the two waves running in opposite directions in relation to the mean electron velocity is not very different. But it appears likely that the medium-like solutions for noise propagation in the preceding accelerated region, if they can be found, would lead to a better understanding of the factors that determine the noise level in a beam and the noise factor in klystrons and traveling-wave tubes.

The discussion of the invalidation of some of the adiabatic solutions by Landau damping is somewhat different in the case of accelerated flow in comparison with the equilibrium states. The perturbation generally produces irreversible changes in the velocity distribution when a significant number of electrons move with velocities equal to or nearly equal to the phase velocity of the wave. In the electron gun the Maxwellian velocity distribution is incomplete, because of the fraction of electrons incapable of passing the potential minimum; consequently waves with phase velocities in this missing range can be expected to have no Landau damping. In this case, therefore, adiabatic solutions may have physical significance, even if their phase velocity is in the thermal range relative to the mean velocity of the electrons.

VI. CONCLUSIONS

The postulate of the existence of adiabatic perturbations traveling as plane compressional waves through an electron gas has produced solutions of Poisson's equation and Boltzmann's transport equation, from which some interesting conclusions can be drawn. In an electron gas at equilibrium between two plane infinite emitters of the same temperature but different emissivity the result was found consistent with the condition of negligible

Landau damping, if the gas was unsaturated, i.e., if one of the emitters was operated strongly temperature-limited. In this case a plasma resonance frequency exists, which is independent of the space coordinates, despite wide variations of the electron density with distance from the emitters. Under all other equilibrium conditions no such plasma resonance frequency exists, and Landau damping will strongly impair all medium-like behavior of the gas in response to compressional disturbances.

Corresponding analysis of an electron gas under condition of accelerated flow has made some progress but no final conclusions have yet been obtained.

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As a first step in the development of a more general theory of propagation and transport of noise by compressional waves in a nonuniform electron gas, this report considers in particular the medium-like propagation in an electron gas in equilibrium with one or two plane emitters. In analogy with acoustic propagation in a neutral gas, adiabatic compressions are tentatively postulated; the solutions are then accepted as physically significant only if they satisfy the conditions for negligible Landau damping.

It is found that the plasma resonance frequency or low-frequency cutoff frequency required to meet these conditions occurs between infinite plane emitters only if one of the emitters is operating strongly temperature-limited. This "unsaturated state" of the gas is precisely specified by the value of an integration constant in the expression for the d-c electric field in the gas. The plasma resonance frequency is found to be constant everywhere in the gas, despite the fact that the electron density varies widely between the emitters.

In no other states, "saturated" or "supersaturated", does such a plasma frequency and the consequent approximately adiabatic propagation exist. In other words, these states show practically no real medium-like behavior. The extension of this analysis to states of accelerated flow is in progress, but corresponding closed-form solutions of the adiabatic wave equation have not yet been found.

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